REPRESENTATION THEORY OF THE HIGHER ORDER PEAK ALGEBRAS

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ABSTRACT. The representation theory (idempotents, quivers, Cartan invariants and Loewy series) of the higher order unital peak algebras is investigated. On the way, we obtain new interpretations and generating functions for the idempotents of descent algebras introduced in [F. Saliola, J. Algebra 320 (2008) 3866.]

1. INTRODUCTION

A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ is an index i such that $\sigma(i) > \sigma(i+1)$. A descent is a *peak* if moreover $i > 1$ and $\sigma(i) > \sigma(i-1)$. The sums of permutations with a given descent set span a subalgebra of the group algebra, the *descent algebra* Σ_n. The *peak algebra* $\mathring{\mathcal{P}}_n$ of $\mathfrak{S}_n$ is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit.

The direct sum of the peak algebras is a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with **Sym**, the Hopf algebra of noncommutative symmetric functions [15]. Actually, in [9] it was shown that most of the results on the peak algebras can be deduced from the case $q = -1$ of a q-identity of [17]. Specializing q to other roots of unity, Krob and the third author introduced and studied *higher order peak algebras* in [18]. Again, these are non-unital.

In [2], it has been shown that the peak algebra of $\mathfrak{S}_n$ can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group B_n. This construction has been extended in [4]. It is shown there that unital versions of the higher order peak algebras can be obtained as homomorphic images of the *Mantaci-Reutenauer algebras* of type B.

Our purpose here is to investigate the representation theory of the unital higher order peak algebras. The classical case has been worked out in [5]. In this reference, idempotents for the peak algebras were obtained from those of the descent algebras of type B constructed in [7].

To deal with the general case, we need a different construction of idempotents. It turns out that the recursive algorithm introduced in [25] for idempotents of descent algebras can be adapted to higher order peak algebras.

In order to achieve this, we need a better understanding of the idempotents generated by the algorithm of [25]. Interpreting them as noncommutative symmetric functions, we find that in type A, these idempotents are associated with a known family of Lie idempotents, the so-called *Zassenhaus idempotents*, by the construction of [17]. We then show that similar Lie idempotents can be defined in type B as well,

which yields a simple generating function in terms of noncommutative symmetric functions of type B.

This being understood, we obtain complete families of orthogonal idempotents for the higher order peak algebras, which can be described either by recurrence relations as in [25] or by generating series of noncommutative symmetric functions.

Finally, we make use of these idempotents to study the quivers, Cartan invariants, and the Loewy series of the unital higher order peak algebras.

2. Notations and background

2.1. **Noncommutative symmetric functions.** We will assume familiarity with the standard notations of the theory of noncommutative symmetric functions [15] and with the main results of [18, 4]. We recall here only a few essential definitions.

The Hopf algebra of noncommutative symmetric functions is denoted by **Sym**, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet A. Linear bases of $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \ldots, i_r)$ of n (we write $I \vDash n$). The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and $S^I = S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I. The *descent set* of I is $\mathrm{Des}(I) = \{i_1, \, i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\}$. The *descent composition* of a permutation $\sigma \in \mathfrak{S}_n$ is the composition $I = D(\sigma)$ of n whose descent set is the descent set of σ.

2.2. **The Mantaci-Reutenauer algebra of type B.** We denote by **MR** the free product $\mathbf{Sym} \star \mathbf{Sym}$ of two copies of the Hopf algebra of noncommutative symmetric functions [20]. That is, **MR** is the free associative algebra on two sequences (S_n) and $(S_{\bar{n}})$ $(n \geq 1)$. We regard the two copies of **Sym** as noncommutative symmetric functions on two auxiliary alphabets: $S_n = S_n(A)$ and $S_{\bar{n}} = S_n(\bar{A})$. We denote by $F \mapsto \bar{F}$ the involutive *anti*automorphism which exchanges S_n and $S_{\bar{n}}$. The bialgebra structure is defined by the requirement that the series

$$(1) \qquad \sigma_1 = \sum_{n \geq 0} S_n \text{ and } \bar{\sigma}_1 = \sum_{n \geq 0} S_{\bar{n}}$$

are grouplike. The internal product of **MR** can be computed from the splitting formula

$$(2) \qquad (f_1 \ldots f_r) * g = \mu_r \cdot (f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g \,,$$

where μ_r is r-fold multiplication, and Δ^r the iterated coproduct with values in the r-th tensor power, and the conditions: σ_1 is neutral, $\bar{\sigma}_1$ is central, and $\bar{\sigma}_1 * \bar{\sigma}_1 = \sigma_1$.

2.3. **Noncommutative symmetric functions of type B.** Noncommutative symmetric functions of type B were introduced in [11] as the right **Sym**-module **BSym** freely generated by another sequence $(\tilde{S}_n)$ $(n \geq 0, \tilde{S}_0 = 1)$ of homogeneous elements, with $\tilde{\sigma}_1$ grouplike. This is a coalgebra, but not an algebra. It is endowed with an

internal product, for which each homogeneous component $\mathbf{BSym}_n$ is anti-isomorphic to the descent algebra of B_n.

It should be noted that with this definition, the restriction of the internal product of $\mathbf{BSym}$ to $\mathbf{Sym}$ is not the internal product of $\mathbf{Sym}$. To remedy this inconvenience, we use a different realization of $\mathbf{BSym}$. We embed $\mathbf{BSym}$ as a sub-coalgebra and sub-$\mathbf{Sym}$-module of $\mathbf{MR}$ as follows. Define, for $F \in \mathbf{Sym}(A)$,

$$(3) \qquad F^\sharp = F(A|\bar{A}) = F(A - q\bar{A})|_{q=-1}$$

called the supersymmetric version, or superization, of F [22]. It is also equal to

$$(4) \qquad F^\sharp = F * \sigma_1^\sharp .$$

Indeed, $\sigma_1^\sharp$ is grouplike, and for $F = S^I$, the splitting formula gives

$$(5) \qquad (S_{i_1} \cdots S_{i_r}) * \sigma_1^\sharp = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\sharp \otimes \cdots \otimes \sigma_1^\sharp)] = S^{I\sharp} .$$

We have

$$(6) \qquad \sigma_1^\sharp = \bar{\lambda}_1 \sigma_1 = \sum \Lambda_{\bar{i}} S_j .$$

The element $\bar{\sigma}_1$ is central for the internal product, and

$$(7) \qquad \bar{\sigma}_1 * F(A, \bar{A}) = F(\bar{A}, A) = F * \bar{\sigma}_1 .$$

The basis element $\tilde{S}^I$ of $\mathbf{BSym}$, where $I = (i_0, i_1, \ldots, i_r)$ is a type B-composition (that is, i_0 may be 0), can be embedded as

$$(8) \qquad \tilde{S}^I = S_{i_0}(A) S^{i_1 i_2 \cdots i_r}(A|\bar{A}) .$$

We will identify $\mathbf{BSym}$ with its image under this embedding.

2.4. **Other notations.** For a partition λ, we denote by $m_i(\lambda)$ the multiplicity of i in λ and set $m_\lambda := \prod_{i \geq 1} m_i(\lambda)!$.

The reverse refinement order on compositions is denoted by $\preceq$. The nonincreasing rearrangement of a composition is denoted by $I\!\downarrow$. The refinement order on partitions is denoted by $\prec_p$: $\lambda \prec_p \mu$ if λ is finer than μ, that is, each part of μ is a sum of parts of λ.

3. Descent algebras of type A

3.1. **Principal idempotents.** In [25], a recursive construction of complete sets of orthogonal idempotents of descent algebras has been described. In [17], one finds a general method for constructing such families from an arbitrary sequence of Lie idempotents, as well as many remarkable families of Lie idempotents. It is therefore natural to investigate whether the resulting idempotents can be derived from a (possibly known) sequence of Lie idempotents. We shall show that it is indeed the case.

Let P_n be the sequence of partitions of n ordered in the following way: first, sort them by decreasing length, then, for each length, order them by reverse lexicographic order. We denote this order by $\leq$. For example,

$$(9) \qquad P_5 = [11111, \ 2111, \ 311, \ 221, \ 41, \ 32, \ 5].$$

Now, start with

$$(10) \qquad\qquad e_{1^n} := \frac{1}{n!} S_1^n,$$

and define by induction

$$(11) \qquad\qquad e_\lambda := \frac{1}{m_\lambda} S^\lambda * \left(S_n - \sum_{\mu < \lambda} e_\mu \right).$$

Theorem 3.1 ([25]). *The family $(e_\lambda)_{\lambda \vdash n}$ forms a complete system of orthogonal idempotents for* $\mathbf{Sym}_n$.

Following [17], define the (left) Zassenhaus idempotents ζ_n by the generating series

$$(12) \qquad\qquad \sigma_1 =: \overset{\leftarrow}{\prod_{k \geq 1}} e^{\zeta_k} = \cdots e^{\zeta_3} e^{\zeta_2} e^{\zeta_1}.$$

For example,

$$(13) \qquad S_1 = \zeta_1 , \quad S_2 = \zeta_2 + \frac{1}{2}\zeta_1^2 , \quad S_3 = \zeta_3 + \zeta_2\zeta_1 + \frac{1}{6}\zeta_1^3,$$

$$(14) \qquad\qquad S_4 = \zeta_4 + \zeta_3\zeta_1 + \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_2\zeta_1^2 + \frac{1}{24}\zeta_1^4,$$

$$(15) \qquad S_5 = \zeta_5 + \zeta_4\zeta_1 + \zeta_3\zeta_2 + \frac{1}{2}\zeta_3\zeta_1^2 + \frac{1}{2}\zeta_2^2\zeta_1 + \frac{1}{6}\zeta_2\zeta_1^3 + \frac{1}{120}\zeta_1^5,$$

$$(16) \qquad S_6 = \zeta_6 + \zeta_5\zeta_1 + \zeta_4\zeta_2 + \frac{1}{2}\zeta_4\zeta_1^2 + \frac{1}{2}\zeta_3^2 + \zeta_3\zeta_2\zeta_1$$
$$+ \frac{1}{6}\zeta_3\zeta_1^3 + \frac{1}{6}\zeta_2^3 + \frac{1}{4}\zeta_2^2\zeta_1^2 + \frac{1}{24}\zeta_2\zeta_1^4 + \frac{1}{720}\zeta_1^6$$

so that

$$(17) \qquad \zeta_1 = S_1 , \quad \zeta_2 = S_2 - \frac{1}{2}S^{11} , \quad \zeta_3 = S_3 - S^{21} + \frac{1}{3}S^{111},$$

$$(18) \qquad \zeta_4 = S_4 - S^{31} - \frac{1}{2}S^{22} + \frac{3}{4}S^{211} + \frac{1}{4}S^{112} - \frac{1}{4}S^{1111},$$

$$(19) \quad \zeta_5 = S_5 - S^{41} - S^{32} + S^{311} + S^{212} - \frac{2}{3}S^{2111} - \frac{1}{3}S^{1112} + \frac{1}{5}S^{11111},$$

$$(20) \quad \zeta_6 = S_6 - S^{51} - S^{42} + S^{411} - \frac{1}{2}S^{33} + \frac{1}{2}S^{321} + S^{312} - \frac{5}{6}S^{3111}$$
$$+ \frac{1}{3}S^{222} - \frac{1}{6}S^{2211} + \frac{1}{2}S^{213} - \frac{1}{2}S^{2121} - \frac{2}{3}S^{2112} + \frac{13}{24}S^{21111}$$
$$- \frac{1}{6}S^{1122} + \frac{1}{12}S^{11211} - \frac{1}{6}S^{1113} + \frac{1}{6}S^{11121} + \frac{5}{24}S^{11112} - \frac{1}{6}S^{111111}.$$

Note that in particular,

$$(21) \qquad S_n = \sum_{\lambda \vdash n} \frac{1}{m_\lambda} \zeta_{\lambda_1} \zeta_{\lambda_2} \dots \zeta_{\lambda_r}.$$

For a composition $I = (i_1, \dots, i_r)$, define as usual $\zeta^I := \zeta_{i_1} \dots \zeta_{i_r}$. Since $\zeta_n \equiv S_n$ modulo smaller terms in the refinement order on compositions, $\zeta^I \equiv S^I$ modulo smaller terms. So the ζ^I family is unitriangular on the basis S^J, so it is a basis of **Sym**.

In the sequel, we shall need a condition for a product $S^I * \zeta^J$ to be zero.

Lemma 3.2. *Let I and J be two compositions of n. Then,*

$$(22) \qquad S^I * \zeta^J = \begin{cases} 0, & \text{if } J\!\downarrow \not\preceq_p I\!\downarrow, \\ m_{I\downarrow} \zeta^I & \text{if } J\!\downarrow = I\!\downarrow, \\ \sum_{K\downarrow = J\downarrow} c_{IJ}^K \zeta^K & \text{otherwise,} \end{cases}$$

where c_{IJ}^K is the number of ways of unshuffling J into $p = \ell(I)$ subwords such that $J^{(l)}$ has sum i_l and whose concatenation $J^{(1)} J^{(2)} \dots J^{(p)}$ is K.

Proof – Since the Zassenhaus idempotents ζ_m are primitive, we have, thanks to the splitting formula (2),

$$(23) \qquad S^I * \zeta^J = \sum_{J^{(1)}, \dots, J^{(p)}} \left(S_{i_1} * \zeta^{J^{(1)}} \right) \cdots \left(S_{i_p} * \zeta^{J^{(p)}} \right)$$

where the sum ranges over all possible ways of decomposing J into p (possibly empty) subwords.

Since $S_{i_j} * \zeta^{J^{(j)}} = 0$ if $J^{(j)}$ is not a composition of i_j, it follows that $S^I * \zeta^J = 0$ if $J\!\downarrow \not\preceq_p I\!\downarrow$. Moreover, if $J\!\downarrow = I\!\downarrow$, then

$$(24) \qquad S^I * \zeta^J = \sum_{\sigma \in \mathfrak{S}_p} \left(S_{i_1} * \zeta_{J_{\sigma(1)}} \right) \cdots \left(S_{i_p} * \zeta_{J_{\sigma(p)}} \right) = m_I \zeta^I.$$

If $J\!\downarrow \prec_p I\!\downarrow$ and $J\!\downarrow \neq I\!\downarrow$, then a term in the r.h.s. of (23) is nonzero iff all $J^{(\ell)}$ are compositions of i_ℓ. In that case, we have

$$(25) \qquad S^I * \zeta^J = \sum_{J^{(1)}, \dots, J^{(p)}} \zeta^{J^{(1)}} \cdots \zeta^{J^{(p)}} = \sum_{J^{(1)}, \dots, J^{(p)}} \zeta^{J^{(1)} J^{(2)} \dots J^{(p)}},$$

whence the last case. ∎

Theorem 3.3. *For all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$,*

$$(26) \qquad e_\lambda = \frac{1}{m_\lambda} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k}.$$

Proof – Let e'_λ be the right-hand side of (26). We will show that these elements satisfy the same induction as the e_λ (Equation (11)).

From Lemma 3.2, we have

$$(27) \qquad S^\lambda * e'_\lambda = m_\lambda e'_\lambda.$$

Now, using (21), we get

$$(28) \qquad m_\lambda e'_\lambda = S^\lambda * \left(S_n - \sum_{\mu \neq \lambda} e'_\mu \right) = S^\lambda * \left(S_n - \sum_{\mu < \lambda} e'_\mu \right),$$

where the last equality follows again from Lemma 3.2. Hence, $e_\lambda = e'_\lambda$. ∎

Note that, thanks to Lemma 3.2, the induction formula for e_λ simplifies to

$$(29) \qquad e_\lambda = \frac{1}{m_\lambda} S^\lambda * \left(S_n - \sum_{\mu \prec_p \lambda} e_\mu \right).$$

3.2. **A basis of idempotents.** As with any sequence of Lie idempotents, we can construct an idempotent basis of $\mathbf{Sym}_n$ from the ζ_n. Here, the principal idempotents e_λ are members of the basis, which leads to a simpler derivation of the representation theory.

We start with a basic lemma, easily derived from the splitting formula (compare [17, Lemma 3.10]). Recall that the radical of $(\mathbf{Sym}_n, *)$ is $\mathcal{R}_n = \mathcal{R} \cap \mathbf{Sym}_n$, where $\mathcal{R}$ is the kernel of the commutative image $\mathbf{Sym} \to Sym$.

Lemma 3.4. *Denote by* $\mathfrak{S}(J)$ *the set of distinct rearrangements of a composition J. Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ be two compositions of n. Then,*

(i) if $\ell(J) < \ell(I)$ then $\zeta^I * \zeta^J = 0$.

(ii) if $\ell(J) > \ell(I)$ then $\zeta^I * \zeta^J \in \mathrm{Vect}\,\langle \zeta^K \, : \, K \in \mathfrak{S}(J) \rangle \cap \mathcal{R}$. More precisely,

$$(30) \qquad \zeta^I * \zeta^J = \sum_{\substack{J_1, \ldots, J_r \\ |J_k| = i_k}} \langle J, \, J_1 \,⧢\, \cdots \,⧢\, J_r \rangle \Gamma_{J_1} \cdots \Gamma_{J_r}$$

where for a composition K of k, $\Gamma_K := \zeta_k * \zeta^K$.

(iii) if $\ell(J) = \ell(I)$, then $\zeta^I * \zeta^J \neq 0$ only for $J \in \mathfrak{S}(I)$, in which case $\zeta^I * \zeta^J = m_I \zeta^I$.

Note that the Γ_K are in the primitive Lie algebra. This follows from the *-multiplicativity of the coproduct: $\Delta(f * g) = \Delta(f) * \Delta(g)$, see [15, Prop. 5.5].

Corollary 3.5. *The elements*

$$(31) \qquad e_I = \frac{1}{m_I} \zeta^I, \quad I \vDash n,$$

are all idempotents and form a basis of $\mathbf{Sym}_n$. This basis contains in particular the principal idempotents e_λ.

3.3. **Cartan invariants.** By (iii) of the lemma, the indecomposable projective module $P_\lambda = \mathbf{Sym}_n * e_\lambda$ contains the e_I for $I \in \mathfrak{S}(\lambda)$. For $I \notin \mathfrak{S}(\lambda)$, (i) and (ii) imply that $e_I * e_\lambda$ is in $\mathrm{Vect}\,\langle \zeta^K \, : \, K \in \mathfrak{S}(\lambda) \rangle$. Hence, this space coincides with P_λ. So, we get immediately an explicit decomposition

$$(32) \qquad \mathbf{Sym}_n = \bigoplus_{\lambda \vdash n} P_\lambda, \qquad P_\lambda = \bigoplus_{I \in \mathfrak{S}(\lambda)} \mathbb{C}\, e_I.$$

The Cartan invariants

(33) $$c_{\lambda\mu} = \dim\left(e_\mu * \mathbf{Sym}_n * e_\lambda\right)$$

are also easily obtained. The above space is spanned by the

(34) $$e_\mu * e_I * e_\lambda = e_\mu * e_I, \quad I \in \mathfrak{S}(\lambda).$$

From (ii) of the lemma, this is the dimension of the space $[S^\mu(L)]_\lambda$, spanned by all symmetrized products of Lie polynomials of degrees $\mu_1, \mu_2, \ldots$ formed from $\zeta_{i_1}, \zeta_{i_2}, \ldots$, hence giving back the classical result of Garsia-Reutenauer [14].

3.4. **Quiver and q-Cartan invariants (Loewy series).** Still relying upon point (ii) of the lemma, we see that $c_{\lambda\mu} = 0$ if λ is not finer than (or equal to) μ, and that if μ is obtained from λ by adding up two parts λ_i, λ_j, $e_\mu * e_I = 0$ if $\lambda_i = \lambda_j$ and is a nonzero element of the radical otherwise.

In [10], it is shown that the powers of the radical for the internal product coincide with the lower central series of $\mathbf{Sym}$ for the external product:

(35) $$\mathcal{R}^{*j} = \gamma^j(\mathbf{Sym})$$

where $\gamma^j(\mathbf{Sym})$ is the ideal generated by the commutators $[\mathbf{Sym}, \gamma^{j-1}(\mathbf{Sym})]$. Hence, for λ finer than μ, $e_\mu * e_I$ is nonzero modulo $\mathcal{R}^{*2}$ iff μ is obtained from λ by summing two distinct parts. And more generally, $e_\mu * e_I$ is in $\mathcal{R}^{*k}$ and nonzero modulo $\mathcal{R}^{*k+1}$ iff $\ell(\lambda) - \ell(\mu) = k$.

Summarizing, we have

Theorem 3.6. [10, 25] *(i) In the quiver of $\mathbf{Sym}_n$, there is an arrow $\lambda \to \mu$ iff μ is obtained from λ by adding two distinct parts.*
(ii) The q-Cartan invariants are given by

(36) $$c_{\lambda\mu}(q) = q^{\ell(\lambda)-\ell(\mu)}$$

if λ is finer than (or equal to) μ, and $c_{\lambda\mu}(q) = 0$ otherwise.

4. Descent algebras of type B

4.1. **Preliminary lemmas on BSym.** We begin by showing that in our realization of **BSym**, Chow's map Θ (see [11, Section 3.4]) corresponds to the left internal product by the reproducing kernel $\sigma_1^\sharp$ of the superization map. Chow's condition $\Theta(\tilde{S}_n) = S_n$ translates into the obvious equality $\sigma_1^\sharp * \sigma_1 = \sigma_1^\sharp$, the second condition $\Theta(S_n(A)) = S_n(2A)$ amounts to

(37) $$\sigma_1^\sharp * \sigma_1^\sharp = (\sigma_1^\sharp)^2,$$

which is an easy consequence of the splitting formula:

(38) $$\sigma_1^\sharp * \sigma_1^\sharp = (\bar{\lambda}_1 \sigma_1) * \sigma_1^\sharp = (\bar{\lambda}_1 * \sigma_1^\sharp)\sigma_1^\sharp = (\sigma_1^\sharp)^2$$

since

(39) $$\bar{\lambda}_1 * \sigma_1^\sharp = (\overline{\lambda}_1 * \sigma_1)(\overline{\lambda}_1 * \overline{\lambda}_1) = \sigma_1^\sharp.$$

Here we used the fact that left $*$-multiplication by $\bar{\lambda}_1$ is an antiautomorphism. Denoting by μ' as in [11] the twisted product

$$\mu'(A \otimes B \otimes C) = (\bar{\lambda}_1 * B)AC, \tag{40}$$

we have:

Lemma 4.1.

$$\sigma_1^\sharp * (FG) = \mu'\left[(\sigma_1^\sharp * F) \otimes \Delta(G)\right]. \tag{41}$$

Proof.

$$
\begin{aligned}
\sigma_1^\sharp * (FG) &= \mu\left[(\bar{\lambda}_1 \otimes \sigma_1) * (\Delta F \Delta G)\right] \\
&= \sum_{(F),(G)} \mu\left[(\bar{\lambda}_1 * F_1 G_1) \otimes F_2 G_2\right] \\
&= \sum(\bar{\lambda}_1 * G_1)((\bar{\lambda}_1 * F_1)F_2)G_2 \\
&= \mu'\left[(\sigma_1^\sharp * F) \otimes \Delta(G)\right].
\end{aligned}
\tag{42}
$$

∎

This is Chow's third condition, which completes the characterization of Θ.

4.2. **Idempotents in BSym.** Define elements $\zeta_n \in \mathbf{BSym}$ by the generating series

$$\sigma_1^\sharp =: \left(e^{\zeta_1} e^{\zeta_2} e^{\zeta_3} \cdots\right)\left(\cdots e^{\zeta_3} e^{\zeta_2} e^{\zeta_1}\right). \tag{43}$$

For example, collecting the terms of weights 1, 2 and 3, respectively, we have

$$S_1^\sharp = 2\zeta_1, \quad S_2^\sharp = 2\zeta_2 + 2\zeta_1^2, \quad S_3^\sharp = 2\zeta_3 + 2\zeta_2\zeta_1 + 2\zeta_1\zeta_2 + \frac{4}{3}\zeta_1^3, \tag{44}$$

so that,

$$\zeta_1 = \frac{1}{2}S_1^\sharp, \quad \zeta_2 = \frac{1}{2}S_2^\sharp - \frac{1}{4}S^{11\sharp}, \quad \zeta_3 = \frac{1}{2}S_3^\sharp - \frac{1}{4}S^{21\sharp} - \frac{1}{4}S^{12\sharp} + \frac{1}{6}S^{111\sharp}. \tag{45}$$

Note that the elements ζ_n are well-defined and that they are primitive. We shall use the notations

$$\left(e^{\zeta_1} e^{\zeta_2} \cdots\right) =: \mathcal{E}^\uparrow(\zeta), \qquad \left(\cdots e^{\zeta_2} e^{\zeta_1}\right) =: \mathcal{E}^\downarrow(\zeta). \tag{46}$$

Next, define elements $\tilde{\zeta}_n \in \mathbf{BSym}$ by the generating series

$$\sigma_1 =: \left(\sum_{n \geq 0} \tilde{\zeta}_n\right)\left(\cdots e^{\zeta_2} e^{\zeta_1}\right) . =: \tilde{\zeta}\mathcal{E}^\downarrow(\zeta). \tag{47}$$

For example,

$$S_1 = \zeta_1 + \tilde{\zeta}_1, \qquad S_2 = \zeta_2 + \frac{1}{2}\zeta_1^2 + \tilde{\zeta}_1\zeta_1 + \tilde{\zeta}_2, \tag{48}$$

$$S_3 = \frac{1}{6}\zeta_1^3 + \zeta_2\zeta_1 + \zeta_3 + \tilde{\zeta}_1\zeta_2 + \frac{1}{2}\tilde{\zeta}_1\zeta_1^2 + \tilde{\zeta}_2\zeta_1 + \tilde{\zeta}_3, \tag{49}$$

so that

$$(50) \qquad \tilde{\zeta}_1 = S_1 - \frac{1}{2}S_1^\sharp, \qquad \tilde{\zeta}_2 = S_2 - \frac{1}{2}S_2^\sharp - \frac{1}{2}S_1S_1^\sharp + \frac{3}{8}S^{11\sharp},$$

$$(51) \qquad \tilde{\zeta}_3 = S_3 - \frac{1}{2}S_2S_1^\sharp - \frac{1}{2}S_1S_2^\sharp + \frac{3}{8}S_1S^{11\sharp} - \frac{1}{2}S_3^\sharp + \frac{1}{4}S^{21\sharp} + \frac{1}{2}S^{12\sharp} - \frac{5}{16}S^{111\sharp}.$$

Since σ_1 is grouplike, and since e^{ζ_n} is grouplike for all $n \geq 1$, the series $\tilde{\zeta}$ is also grouplike.

   The next two lemmas describe some properties of the elements $\tilde{\zeta}_n$ and ζ_n.

Lemma 4.2. *The ordered exponentials are exchanged as follows:*

$$(52) \qquad \overline{\lambda}_1 * \mathcal{E}^\downarrow(\zeta) = \mathcal{E}^\uparrow(\zeta).$$

In particular, $\overline{\lambda}_1 * \zeta_i = \zeta_i$ for all $i \geq 0$.

Proof – $\overline{\lambda}_1 * \cdot$ is an antiautomorphism, so the left-hand side is

$$(53) \qquad (\overline{\lambda}_1 * e^{\zeta_1})(\overline{\lambda}_1 * e^{\zeta_2})\dots$$

Taking into account (39), and recalling that (43) characterizes the ζ_i, we see that if we set

$$(54) \qquad \zeta_i' := \overline{\lambda}_1 * \zeta_i,$$

then

$$(55) \qquad \sigma_1^\sharp = \left(e^{\zeta_1'} e^{\zeta_2'} \cdots \right) \left(\cdots e^{\zeta_2'} e^{\zeta_1'} \right),$$

so that $\zeta_i' = \zeta_i$. ∎

Lemma 4.3. *For all $n \geq 1$,*

$$(56) \qquad \sigma_1^\sharp * \tilde{\zeta}_n = 0.$$

Proof – By definition,

$$(57) \qquad
\begin{aligned}
\mathcal{E}^\uparrow(\zeta)\mathcal{E}^\downarrow(\zeta) = \sigma_1^\sharp &= \sigma_1^\sharp * \sigma_1 \\
&= \sigma_1^\sharp * \left[\tilde{\zeta}\mathcal{E}^\downarrow(\zeta) \right] \\
&= (\overline{\lambda}_1\sigma_1) * (\tilde{\zeta}\mathcal{E}^\downarrow(\zeta)) \\
&= \mu\left[(\overline{\lambda}_1 \otimes \sigma_1) * (\tilde{\zeta}\mathcal{E}^\downarrow(\zeta) \otimes \tilde{\zeta}\mathcal{E}^\downarrow(\zeta)) \right] \\
&= (\overline{\lambda}_1 * \tilde{\zeta}\mathcal{E}^\downarrow(\zeta))\ (\tilde{\zeta}\mathcal{E}^\downarrow(\zeta)) \\
&= (\overline{\lambda}_1 * \mathcal{E}^\downarrow(\zeta))\ (\overline{\lambda}_1 * \tilde{\zeta})\ \tilde{\zeta}\ \mathcal{E}^\downarrow(\zeta) \\
&= \mathcal{E}^\uparrow(\zeta)\ (\overline{\lambda}_1 * \tilde{\zeta})\ (\sigma_1 * \tilde{\zeta})\ \mathcal{E}^\downarrow(\zeta) \\
&= \mathcal{E}^\uparrow(\zeta)\ (\sigma_1^\sharp * \tilde{\zeta})\ \mathcal{E}^\downarrow(\zeta)
\end{aligned}$$

so that $\sigma_1^\sharp * \tilde{\zeta} = 1$. ∎

Lemma 4.4. *For all $n \geq 1$,*

$$(58) \qquad\qquad \sigma_1^\sharp * \zeta_n = 2\zeta_n.$$

Proof – We have

$$(59) \qquad \begin{aligned} \sigma_1^\sharp * \zeta_n &= (\overline{\lambda}_1 \sigma_1) * \zeta_n \\ &= \mu \left[(\overline{\lambda}_1 \otimes \sigma_1) * (\zeta_n \otimes 1 + 1 \otimes \zeta_n) \right] \\ &= 2\zeta_n. \end{aligned}$$

$\blacksquare$

Proposition 4.5. *Let $I = (i_0, \ldots, i_p)$ be a B-composition of n and let $\lambda = (\lambda_0, \ldots, \lambda_k)$ be a B-partition of n.*

$$(60) \qquad \tilde{S}^I * \tilde{\zeta}_{\lambda_0} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k} = \begin{cases} 0, & \text{if } \lambda \not\preceq I\downarrow, \\ \left(2^p \prod_{j \geq 1} m_j! \right) \tilde{\zeta}_{i_0} \zeta_{i_1} \cdots \zeta_{i_p} & \text{if } \lambda = I\downarrow, \end{cases}$$

where m_j is the multiplicity of j in $(i_1, i_2, \ldots, i_p)$ (not counting i_0!).

Proof – The splitting formula yields

$$(61) \qquad \begin{aligned} \tilde{S}^I * \tilde{\zeta}_{\lambda_0} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k} &= \mu_p \left[(S_{i_0} \otimes S_{i_1}^\sharp \otimes \cdots \otimes S_{i_p}^\sharp) * \sum \tilde{\zeta}_{\alpha_0}^\sharp \zeta^{\alpha^{(0)}} \otimes \cdots \otimes \tilde{\zeta}_{\alpha_p}^\sharp \zeta^{\alpha^{(p)}} \right] \\ &= \sum (S_{i_0} * \tilde{\zeta}_{\alpha_0}^\sharp \zeta^{\alpha^{(0)}})(S_{i_1}^\sharp * \tilde{\zeta}_{\alpha_1}^\sharp \zeta^{\alpha^{(1)}}) \cdots (S_{i_p}^\sharp * \tilde{\zeta}_{\alpha_p}^\sharp \zeta^{\alpha^{(p)}}). \end{aligned}$$

By Equation (4) and Lemma 4.3, a summand is zero if any $\alpha_i > 0$ for $i \geq 1$, so that

$$(62) \qquad \tilde{S}^I * \tilde{\zeta}_{\lambda_0} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k} = \sum (S_{i_0} * \tilde{\zeta}_{\alpha_0}^\sharp \zeta^{\alpha^{(0)}})(S_{i_1}^\sharp * \zeta^{\alpha^{(1)}}) \ldots (S_{i_r}^\sharp * \zeta^{\alpha^{(r)}}).$$

If a term is non-zero in this equation, then $\lambda \preceq I\downarrow$. This proves the first case. By Equation (4) and Lemma 4.4, $S_i^\sharp * \zeta_i = 2\zeta_i$, which proves the second case. $\blacksquare$

We are now in a position to give an explicit formula for the idempotents of [25].

Theorem 4.6. *For all B-partitions $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k)$ of n, define elements $e_\lambda \in \mathbf{BSym}_n$ recursively by the formula*

$$(63) \qquad\qquad e_\lambda = \frac{1}{2^k \prod_j m_j!} \tilde{S}^\lambda * \left(S_n - \sum_{\mu < \lambda} e_\mu \right),$$

where m_j is the multiplicity of j in $(\lambda_1, \ldots, \lambda_k)$ (not counting λ_0!). Then

$$(64) \qquad\qquad e_\lambda = \frac{1}{\prod_j m_j!} \tilde{\zeta}_{\lambda_0} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k}.$$

Proof – Let e_λ' be the right-hand side of the above equation. By Proposition 4.5,

$$(65) \qquad \tilde{S}^\lambda * e_\lambda' = \tilde{S}^\lambda * \frac{1}{\prod_j m_j!} \tilde{\zeta}_{\lambda_0} \zeta_{\lambda_1} \cdots \zeta_{\lambda_k} = \left(2^k \prod_j m_j! \right) e_\lambda'.$$

By Equation (47), $S_n = \sum e'_\lambda$, where the sum ranges over all B-partitions λ of n. Together with the above and Proposition 4.5, we have

$$(66) \qquad \left(2^k \prod_j m_j!\right) e'_\lambda = \tilde{S}^\lambda * e'_\lambda = \tilde{S}^\lambda * \left(S_n - \sum_{\mu \neq \lambda} e'_\mu\right) = \tilde{S}^\lambda * \left(S_n - \sum_{\mu < \lambda} e'_\mu\right).$$

Since the e'_λ satisfy the same induction as the e_λ, they are equal. ∎

5. IDEMPOTENTS IN THE HIGHER ORDER PEAK ALGEBRAS

Let q be a primitive r-th root of unity. We denote by θ_q the endomorphism of **Sym** defined by

$$(67) \qquad \tilde{f} = \theta_q(f) = f((1-q)A) = f(A) * \sigma_1((1-q)A).$$

We denote by $\mathring{\mathcal{P}}^{(r)}$ the image of θ_q and by $\mathcal{P}^{(r)}$ the right $\mathring{\mathcal{P}}^{(r)}$-module generated by the S_n for $n \geq 0$. Note that $\mathring{\mathcal{P}}^{(r)}$ is by definition a left $*$-ideal of **Sym**. For $r = 2$, it is the classical peak ideal, and $\mathcal{P}^{(2)}$ is the unital peak algebra. For general r, $\mathring{\mathcal{P}}^{(r)}$ is the higher order peak algebra of [18] and $\mathcal{P}^{(r)}$ is its unital extension defined in [4]. These objects depend only on r, and not on the choice of the primitive root of unity. Bases of $\mathcal{P}^{(r)}$ can be labeled by r-peak compositions $I = (i_0; i_1, \ldots, i_p)$, with at most one part i_0 divisible by r.

5.1. **The radical.** By definition, $\mathcal{P}_n^{(r)}$ is a $*$-subalgebra of **Sym**$_n$. The radical of **Sym**$_n$ consists of those elements whose commutative image is zero (see [17], Lemma 3.10). The radical of $\mathcal{P}_n^{(r)}$ is therefore spanned by the $S_{i_0} \cdot \theta_q(S^I - S^{I'})$ such that I' is a permutation of I. Indeed, the quotient of $\mathcal{P}_n^{(r)}$ by the span of those elements is a semi-simple commutative algebra, the $*$-subalgebra of Sym_n spanned by the p_λ ($\lambda \vdash n$) such that at most one part of λ is multiple of r. This special part will be denoted by λ_0.

We denote by $P_n^{(r)}$ the subset of partitions of n with at most one part divisible by r. The simple $\mathcal{P}_n^{(r)}$-modules, and the principal idempotents, can therefore be labeled by $P_n^{(r)}$.

5.2. **An induction for the idempotents.** Define a total order $<$ on $P_n^{(r)}$ as follows: sort the partitions by decreasing length, and sort partitions of the same length by reverse lexicographic order. For example,

$$(68) \qquad\qquad P_5^{(2)} = [11111,\ 2111,\ 311,\ 41,\ 23,\ 5],$$

$$(69) \qquad P_7^{(2)} = [1111111,\ 211111,\ 31111,\ 4111,\ 2311,\ 511,\ 331,\ 61,\ 43,\ 25,\ 7].$$

Now, set

$$(70) \qquad\qquad\qquad e_{1^n}^{(r)} := \frac{1}{n!} S_1^n,$$

and define by induction

$$(71) \qquad e_\lambda^{(r)} := \frac{1}{m_\lambda} \, T^\lambda * \left(S_n - \sum_{\mu < \lambda} e_\mu^{(r)} \right)$$

where $T_m = R_m$ if $r \mid m$, $T_m = R_{r^i j}$ if $r \nmid m$ and $m = ir + j$ with $0 < j < r$, and $T^\lambda = T_{\lambda_0} T_{\lambda_1} \cdots T_{\lambda_p}$ for $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_p) \in P_n^{(r)}$. It follows from [18, Cor. 3.17] and from the definition of $\mathcal{P}^{(r)}$ that $T^\lambda \in \mathcal{P}^{(r)}$.

We want to prove that $(e_\lambda^{(r)})_{\lambda \in P_n^{(r)}}$ is a complete system of orthogonal idempotents for $\mathcal{P}^{(r)}$.

To this aim, we introduce the sequence of (left) Zassenhaus idempotents of level r $\zeta_n^{(r)}$ as the unique solution of the equation

$$(72) \qquad \sigma_1 = \left(\sum_{p \geq 0} \zeta_{pr}^{(r)} \right) \overleftarrow{\prod_{i \geq 1, \ r \nmid i}} e^{\zeta_i^{(r)}}.$$

Note that $\zeta_n^{(r)} = \zeta_n$ for $n < 2r$.

For example, for $r = 2$,

$$(73) \qquad \zeta_1^{(2)} = S_1 \ ; \ \zeta_2^{(2)} = S_2 - \frac{1}{2} S^{11} \ ; \ \zeta_3^{(2)} = S_3 - S^{21} + \frac{1}{3} S^{111},$$

$$(74) \qquad \zeta_4^{(2)} = S_4 - S^{31} + \frac{1}{2} S^{211} - \frac{1}{8} S^{1111},$$

$$(75) \quad \zeta_5^{(2)} = S_5 - S^{41} + \frac{1}{2} S^{311} - S^{23} + S^{221} - \frac{1}{2} S^{2111} + \frac{1}{2} S^{113} - \frac{1}{2} S^{1121} + \frac{1}{5} S^{11111}.$$

And for $r = 3$,

$$(76) \qquad \zeta_1^{(3)} = S_1 \ ; \ \zeta_2^{(3)} = S_2 - \frac{1}{2} S^{11} \ ; \ \zeta_3^{(3)} = S_3 - S^{21} + \frac{1}{3} S^{111},$$

$$(77) \qquad \zeta_4^{(3)} = S_4 - S^{31} - \frac{1}{2} S^{22} + \frac{3}{4} S^{211} + \frac{1}{4} S^{112} - \frac{1}{4} S^{1111},$$

$$(78) \qquad \zeta_5^{(3)} = S_5 - S^{41} - S^{32} + S^{311} + S^{212} - \frac{2}{3} S^{2111} - \frac{1}{3} S^{1112} + \frac{1}{5} S^{11111},$$

$$\zeta_6^{(3)} = S_6 - S^{51} - S^{42} + S^{411} + S^{312} - \frac{2}{3} S^{3111} + \frac{1}{3} S^{222} - \frac{1}{6} S^{2211}$$
$$(79) \qquad - \frac{2}{3} S^{2112} + \frac{3}{8} S^{21111} - \frac{1}{6} S^{1122} + \frac{1}{12} S^{11211} + \frac{5}{24} S^{11112} - \frac{1}{9} S^{111111}.$$

Define now for $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_k) \in P_n^{(r)}$,

$$(80) \qquad e'^{(r)}_\lambda := \frac{1}{m_\lambda} \, \zeta_{\lambda_0}^{(r)} \zeta_{\lambda_1}^{(r)} \cdots \zeta_{\lambda_k}^{(r)}.$$

We will show that $e'^{(r)}_\lambda = e_\lambda^{(r)}$ for all $\lambda \in P_n^{(r)}$. We begin with two lemmas.

Lemma 5.1.

$$(81) \qquad \Delta\left(\zeta_n^{(r)}\right) = \begin{cases} 1 \otimes \zeta_n^{(r)} + \zeta_n^{(r)} \otimes 1, & \text{if } r \nmid n, \\ \sum_{i=0}^{n/r} \zeta_{ir}^{(r)} \otimes \zeta_{n-ir}^{(r)}, & \text{if } r \mid n. \end{cases}$$

Proof – This means that the $\zeta_n^{(r)}$ are primitive if $r \mid n$ and that the generating series $\sum_{p \geq 0} \zeta_{rp}^{(r)}$ is grouplike. If we define new elements Y_p by

$$(82) \qquad \sigma_1 =: \overrightarrow{\prod_p} e^{Y_{rp}} \overleftarrow{\prod_{r \nmid i}} e^{Y_i},$$

the standard argument showing that the Zassenhaus elements are primitive shows as well that all the Y_i are primitive. Now identify the first product in the right-hand side with the generating series $\sum_{p \geq 0} \zeta_{rp}^{(r)}$. Then $\zeta_i^{(r)} = Y_i$ if $r \nmid i$. Since the exponential of a primitive element is grouplike and a product of grouplike series is group-like, both products in the right-hand side above are grouplike. By identification, $\sum_{p \geq 0} \zeta_{rp}^{(r)}$ is grouplike. ∎

Lemma 5.2. *Let* $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_k) \in P_n^{(r)}$ *and* $I = (i_0, i_1, \ldots, i_p)$ *be an r-peak composition of n. Then,*

$$(83) \qquad T^I * \zeta_{\lambda_0}^{(r)} \zeta_{\lambda_1}^{(r)} \cdots \zeta_{\lambda_k}^{(r)} = \begin{cases} 0, & \text{if } I{\downarrow} < \lambda, \\ m_I \ \zeta_{i_0}^{(r)} \zeta_{i_1}^{(r)} \cdots \zeta_{i_k}^{(r)}, & \text{if } I{\downarrow} = \lambda. \end{cases}$$

Proof – To simplify the notation, we let $\zeta^{(r)\lambda} = \zeta_{\lambda_0}^{(r)} \zeta_{\lambda_1}^{(r)} \cdots \zeta_{\lambda_k}^{(r)}$ for $\lambda \in P_n^{(r)}$. If $F^I = F_{i_1} \cdots F_{i_p}$ with each $F_{i_j} \in \mathbf{Sym}_{i_j}$, the splitting formula and (81) yield

$$(84) \qquad F^I * \zeta^{(r)\lambda} = \sum_{\substack{\lambda^{(1)} \vee \cdots \vee \lambda^{(p)} = (\lambda_1, \ldots, \lambda_k) \\ ra_1 + \cdots + ra_p = \lambda_0}} \prod_{j=1}^{p} \left(F_{i_j} * \zeta_{ra_j}^{(r)} \zeta^{(r)\lambda^{(j)}} \right)$$

where λ_0 is the part (possibly 0) of λ that is divisible by r, and where $\alpha \vee \beta$ denotes the partition obtained by reordering the concatenation of the partitions α and β.

Observe that since at most one i_j is divisible by r, a product in the above summation is 0 if at least two of the partitions $\lambda^{(1)}, \ldots, \lambda^{(p)}$ are empty. If $\ell(I) > \ell(\lambda)$, then $k \leq p - 2$, so this hypothesis is always satisfied. Thus,

$$(85) \qquad F^I * \zeta^{(r)\lambda} = 0 \text{ if } \ell(I) > \ell(\lambda).$$

Suppose $I{\downarrow} \leq \lambda$. Then, by definition of the order, $\ell(I) \geq \ell(\lambda)$. Hence, for I such that $I{\downarrow} \leq \lambda$ and $\ell(I) > \ell(\lambda)$, the result follows by taking $F^I = T^I$ in (85). So suppose instead that $\ell(I) = \ell(\lambda)$. By definition, $T^I = T_{i_1} \cdots T_{i_p}$, where $T_m = R_m$ if $r \mid m$ and $T_m = R_{r^i j}$ if $m = ir + j$ with $0 < j < r$. Since R_J can be written as a linear combination of S^K for which J is a refinement of K, it follows that T^I is equal to S^I plus a linear combination of S^K with $\ell(K) > \ell(I)$. By taking $F^I = S^K$ in (85), it follows that $T^I * \zeta^{(r)\lambda} = S^I * \zeta^{(r)\lambda}$.

It remains to show that, for I and λ of the same length, $S^I * \zeta^{(r)\lambda} = m_I \zeta^{(r)\lambda}$ if $I\!\downarrow = \lambda$ and is 0 otherwise. If λ contains no part divisible by r, then it follows from (84) that if $S^I * \zeta^{(r)\lambda} \neq 0$, we must have $I\!\downarrow = \lambda$, in which case $S^I * \zeta^{(r)\lambda} = m_I \zeta^{(r)\lambda}$.

Suppose instead that λ contains a part that is divisible by r. Then each decomposition $(\lambda^{(1)}, \ldots, \lambda^{(p)})$ in (84) contains at least one $\lambda^{(j)} = \emptyset$. Thus, if I contains no part that is divisible by r, then $S^I * \zeta^{(r)\lambda} = 0$. Otherwise, the part of I that is divisible by r is bounded by λ_0. This implies that $\lambda \leq I\!\downarrow$. Since we began by assuming that $I\!\downarrow \leq \lambda$, it follows that $I\!\downarrow = \lambda$, and the result follows from (84) as before. ∎

Theorem 5.3. *For all partitions* $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_k) \in P_n^{(r)}$,

$$(86) \qquad e_\lambda^{(r)} = e'^{(r)}_\lambda := \frac{1}{m_\lambda} \zeta_{\lambda_0}^{(r)} \zeta_{\lambda_1}^{(r)} \cdots \zeta_{\lambda_k}^{(r)}.$$

Proof – From the definition of $\zeta_m^{(r)}$, it follows that $S_n = \sum_{\lambda \in P_n^{(r)}} e'^{(r)}_\lambda$. Hence, by Lemma 5.2,

$$m_\lambda e'^{(r)}_\lambda = T^\lambda * e'_\lambda = T^\lambda * \left(S_n - \sum_{\mu \neq \lambda} e'^{(r)}_\mu \right) = T^\lambda * \left(S_n - \sum_{\mu < \lambda} e'^{(r)}_\mu \right).$$

Thus, the elements $e'^{(r)}_\lambda$ and $e_\lambda^{(r)}$ satisfy the same induction equation (71). ∎

Theorem 5.4. *The family* $(e_\lambda^{(r)})_{\lambda \in P_n^{(r)}}$ *forms a complete system of orthogonal idempotents for* $\mathcal{P}_n^{(r)}$.

Proof. By construction, the $e_\lambda^{(r)}$ are in $\mathcal{P}_n^{(r)}$. Their identification with the $e'^{(r)}_\lambda$ shows that they are linearly independent, and (82) shows that they are orthogonal idempotents. Indeed, the Y_i are Lie idempotents of **Sym**, and if we write partitions μ of n as $(\alpha; \beta)$, where the α_i are the parts divisible by r in increasing order and the β_i the other parts in decreasing order, then

$$(87) \qquad Y_\mu = \frac{1}{m_\alpha m_\beta} Y^\alpha Y^\beta$$

is a complete set of orthogonal idempotents of $\mathbf{Sym}_n$. Hence, the $e_\lambda^{(r)}$, which are disjoint sums of the Y_μ, are orthogonal idempotents. ∎

5.3. **Cartan invariants.** Recall from subsection 5.1 that the indecomposable projective modules of $\mathcal{P}_n^{(r)}$ can be labelled by $\mu \in P_n^{(r)}$. For $\mu = (\mu_0; \mu_1, \ldots, \mu_p) \in P_n^{(r)}$, where μ_0 is is divisible by r, let $\bar{\mu} = (\mu_1, \ldots, \mu_p)$, and write $\mu = (\mu_0; \bar{\mu})$.

The description of the principal orthogonal idempotents in the previous subsection shows that the dimension of the indecomposable projective module labeled by μ is equal to the number of distinct permutations of $\bar{\mu}$. We make the following conjecture:

Conjecture 5.5. *The Cartan invariant* $\dim(e_\nu^{(r)} * \mathcal{P}_n^{(r)} * e_\mu^{(r)})$ *is the number of permutations* $\bar{I}$ *of* $\bar{\mu}$ *for which the following algorithm produces* $\bar{\nu}$.

(1) *Compute the standardization $\tau = \mathrm{Std}(\bar{I})$.*
(2) *Replace the elements of the cycles of τ by the corresponding values in $\bar{I}$.*
(3) *Take the sums of the values in each cycle, discarding those that are multiples of r.*
(4) *The partition obtained by reordering these sums is $\bar{\nu}$.*

We will prove this for the classical case ($r = 2$) in the next section. We will also prove that the matrix of the Cartan invariants of $\mathcal{P}_n^{(2)}$ is obtained from that of $\mathbf{BSym}_n$ by merely selecting the rows and columns labelled by 2-peak partitions $P_n^{(2)}$. (That is, we select for the type B partitions $(\alpha_0; \alpha_1, \ldots, \alpha_p)$ such that α_0 is even and $\alpha_1, \ldots, \alpha_p$ are odd.) We know of no such simple description, even conjectural, for the general case.

6. The classical peak algebras ($r = 2$)

In this section, we restrict our attention to the classical peak algebra $\mathcal{P}_n := \mathcal{P}_n^{(2)}$. We will compute the q-Cartan matrix of $\mathcal{P}_n$, thus determining the quiver of $\mathcal{P}_n$ and proving the conjecture at the end of the previous section for $r = 2$.

Recall that for type B compositions or peak compositions $I = (i_0; i_1, \ldots, i_p)$, we define $\bar{I} = (i_1, \ldots, i_p)$.

6.1. q-Cartan invariants and quiver.

Let $\mathfrak{proj}(F) = F|_{\bar{A}=A}$ denote the projection of an arbitrary element $F \in \mathbf{MR}$ onto $\mathbf{Sym}$. We will show that the projection of $\mathbf{BSym}$ is $\mathcal{P}$, and also that $\mathcal{P}$ can be identified with a subalgebra of $\mathbf{BSym}$. This identification allows us to use known results about the quiver and q-Cartan invariants of $\mathbf{BSym}$ to study $\mathcal{P}$.

Lemma 6.1.

$$(88) \quad \mathfrak{proj}(\zeta_n) = \begin{cases} \zeta_n^{(2)}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad \mathfrak{proj}(\tilde{\zeta}_n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \zeta_n^{(2)}, & \text{if } n \text{ is even.} \end{cases}$$

In particular, if I is a type B composition, then

$$(89) \quad \mathfrak{proj}(e_I) = \begin{cases} e_I^{(2)}, & \text{if } I \text{ is a 2-peak composition,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof – Recall that the elements $\zeta_n, \tilde{\zeta}_n \in \mathbf{BSym}$ are defined as the coefficients of t^n in the following two series, respectively,

$$(90) \quad \sigma_t^{\sharp} = \left(e^{\zeta_1 t} e^{\zeta_2 t^2} e^{\zeta_3 t^3} \cdots \right) \left(\cdots e^{\zeta_3 t^3} e^{\zeta_2 t^2} e^{\zeta_1 t} \right),$$

$$(91) \quad \sigma_t = \left(\sum_{m \geq 0} \tilde{\zeta}_m t^m \right) \left(\cdots e^{\zeta_3 t^3} e^{\zeta_2 t^2} e^{\zeta_1 t} \right),$$

where

$$(92) \quad \sigma_t^{\sharp} := \sum_{n \geq 0} S_n^{\sharp} t^n, \qquad \sigma_t := \sum_{n \geq 0} S_n t^n.$$

The projection of $\sigma_t^{\sharp}$ is

$$(93) \qquad \mathfrak{proj}(\sigma_t^{\sharp}) = \mathfrak{proj}(\bar{\lambda}_t \sigma_t) = \lambda_t \sigma_t,$$

which satisfies

$$(94) \qquad \mathfrak{proj}(\sigma_{-t}^{\sharp}) = \lambda_{-t}\sigma_{-t} = (\lambda_t \sigma_t)^{-1} = \mathfrak{proj}(\sigma_t^{\sharp})^{-1}.$$

Combined with Equation (90), this yields the identity

$$(95) \quad \left(e^{-\mathfrak{proj}(\zeta_1)t} e^{\mathfrak{proj}(\zeta_2)t^2} e^{-\mathfrak{proj}(\zeta_3)t^3} \cdots \right) \left(\cdots e^{-\mathfrak{proj}(\zeta_3)t^3} e^{\mathfrak{proj}(\zeta_2)t^2} e^{-\mathfrak{proj}(\zeta_1)t} \right)$$

$$= \left(e^{-\mathfrak{proj}(\zeta_1)t} e^{-\mathfrak{proj}(\zeta_2)t^2} e^{-\mathfrak{proj}(\zeta_3)t^3} \cdots \right) \left(\cdots e^{-\mathfrak{proj}(\zeta_3)t^3} e^{-\mathfrak{proj}(\zeta_2)t^2} e^{-\mathfrak{proj}(\zeta_1)t} \right),$$

from which it follows that $\mathfrak{proj}(\zeta_{2m}) = 0$ for all $m \geq 1$.

From Equations (90) and (91), we have that

$$(96) \qquad \sigma_t = \left(\sum_{m \geq 0} \tilde{\zeta}_m t^m \right) \left(\cdots e^{\zeta_3 t^3} e^{\zeta_2 t^2} e^{\zeta_1 t} \right)$$

$$(97) \qquad = \bar{\sigma}_{-t} \left(e^{\zeta_1 t} e^{\zeta_2 t^2} e^{\zeta_3 t^3} \cdots \right) \left(\cdots e^{\zeta_3 t^3} e^{\zeta_2 t^2} e^{\zeta_1 t} \right),$$

and so a generating series for the $\tilde{\zeta}_m$ is given by

$$(98) \qquad \tilde{\zeta}(t) := \sum_{m \geq 0} \tilde{\zeta}_m t^m = \bar{\sigma}_{-t} \left(e^{\zeta_1 t} e^{\zeta_2 t^2} e^{\zeta_3 t^3} \cdots \right).$$

Denoting by $\mathfrak{proj}(\tilde{\zeta})(t)$ the projection of $\tilde{\zeta}(t)$, and remembering that $\mathfrak{proj}(\zeta_{2m}) = 0$,

$$(99) \qquad \mathfrak{proj}(\tilde{\zeta})(-t) = \sigma_t \left(e^{-\mathfrak{proj}(\zeta_1)t} e^{\mathfrak{proj}(\zeta_2)t^2} e^{-\mathfrak{proj}(\zeta_3)t^3} \cdots \right) = \mathfrak{proj}(\tilde{\zeta})(t),$$

where the last equality follows from Equation (91). Thus, $\mathfrak{proj}(\tilde{\zeta})(t)$ is even, which implies that $\mathfrak{proj}(\tilde{\zeta}_{2m-1}) = 0$ for $m \geq 1$.

Next, note that the image of the generating series in Equation (91) is precisely the generating series of the elements $\zeta_n^{(r)}$ given in Equation (72). Since this later generating series uniquely defines the elements $\zeta_n^{(2)}$, it follows that

$$(100) \qquad \mathfrak{proj}(\tilde{\zeta}_{2m}) = \zeta_{2m}^{(2)} \quad \text{and} \quad \mathfrak{proj}(\zeta_{2m+1}) = \zeta_{2m+1}^{(2)}.$$

For the last assertion, recall that Equation (64) defines $e_I \in \mathbf{BSym}$ for any type B composition as

$$(101) \qquad e_I = \frac{1}{m_{\bar{I}}} \tilde{\zeta}_{i_0} \zeta_{i_1} \cdots \zeta_{i_p}.$$

Hence, if I is not a 2-peak composition (that is, if i_0 is not even or if any of $i_1, \ldots, i_p$ are not odd), then $\mathfrak{proj}(e_I) = 0$. And if I is a 2-peak composition, then

$$(102) \qquad \mathfrak{proj}(e_I) = \frac{1}{m_{\bar{I}}} \zeta_{i_0}^{(r)} \zeta_{i_1}^{(r)} \cdots \zeta_{i_p}^{(r)} = e_I^{(2)}$$

where the last equality is just the definition of $e_I^{(2)} \in \mathcal{P}$ (see Equation (86)).   ∎

Using this result we can prove that the peak algebra $\mathcal{P}_n$ is both a quotient and a subalgebra of $\mathbf{BSym}_n$.

Theorem 6.2. *(i) The projection of* $\mathbf{BSym}$ *onto* $\mathbf{Sym}$ *is* $\mathfrak{proj}(\mathbf{BSym}) = \mathcal{P}$.
(ii) $\mathcal{P}_n$ *is isomorphic to the subalgebra* $\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n$ *of* $\mathbf{BSym}_n$, *where*

$$(103) \qquad \varepsilon_n = \sum_{\lambda \in P_n^{(2)}} e_\lambda \in \mathbf{BSym}_n.$$

Proof – From Lemma 6.1, the basis $(e_I)_I$ of $\mathbf{BSym}$, where I is a type B composition, maps onto the basis $(e_J^{(2)})_J$ of $\mathcal{P}$, where J is a 2-peak composition. This proves (i).

Since $\mathfrak{proj}(e_\lambda) = e_\lambda^{(2)}$ for 2-peak partitions λ,

$$(104) \qquad \mathfrak{proj}(\varepsilon_n) = \sum_{\lambda \in P_n^{(2)}} e_\lambda^{(2)} = 1 \in \mathcal{P}_n.$$

Hence, $\mathfrak{proj}(\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n) = \mathcal{P}_n$. We now only need to prove that the two algebras have the same dimension.

For any 2-peak partition λ, the dimension of $\mathbf{BSym}_n * e_\lambda$ is the number of rearrangements of $\bar{\lambda}$, which is also the dimension of $\mathcal{P}_n * e_\lambda^{(2)}$ (these statements follow from the facts that the elements e_I and $e_I^{(2)}$, one for each 2-peak composition I such that $\bar{I}{\downarrow} = \bar{\lambda}$, form bases of $\mathbf{BSym}_n * e_\lambda$ and $\mathcal{P}_n * e_\lambda^{(2)}$, respectively). This implies that

$$(105) \quad \dim\left(\mathcal{P}_n\right) = \dim\left(\mathcal{P}_n * \sum_{\lambda \in P_n^{(2)}} e_\lambda^{(2)}\right) = \dim(\mathbf{BSym}_n * \varepsilon_n)$$

$$(106) \qquad \geq \dim(\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n) \geq \dim(\mathfrak{proj}(\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n)) = \dim(\mathcal{P}_n),$$

so that $\dim(\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n) = \dim(\mathcal{P}_n)$. ∎

Corollary 6.3. *Let us label the vertices of the quiver* Q_n *of* $\mathcal{P}_n$ *by odd partitions* $\bar{\mu}$ *of* $n, n-2, n-4, \ldots$. *There are exactly* $m > 0$ *arrows from* $\bar{\mu}$ *to* $\bar{\nu}$ *in* Q_n *if and only if* $\bar{\nu}$ *is obtained from* $\bar{\mu}$ *by: deleting two unequal parts (in which case* $m = 1$); *or merging three parts, of which at most two are equal (if no two of the merged parts are equal, then* $m = 2$; *if exactly two are equal, then* $m = 1$).

Proof – By Theorem 6.2, we can identify $\mathcal{P}_n$ with the subalgebra $\varepsilon_n * \mathbf{BSym}_n * \varepsilon_n$ of $\mathbf{BSym}$. Since ε_n is a sum of primitive orthogonal idempotents of $\mathbf{BSym}_n$, it follows that the quiver Q_n of $\mathcal{P}_n$ is obtained from the quiver of $\mathbf{BSym}_n$ by restricting to the vertices labelled by 2-peak partitions λ. The result follows immediately as the quiver of $\mathbf{BSym}_n$ has already been computed [25, Theorem 9.1]. ∎

From these results we can easily derive the results mentioned at the end of the previous section for the $r = 2$ case.

Theorem 6.4.
    (i) The Cartan matrix of $\mathcal{P}_n$ *is obtained from that of* $\mathbf{BSym}_n$ *by selecting the rows and columns labelled by the 2-peak partitions* $P_n^{(2)}$.

(ii) For $r = 2$, Conjecture 5.5 is true. Moreover, continuing with the notation of Conjecture 5.5, I contributes $q^{\frac{1}{2}(l(\bar{\mu}) - l(\bar{\nu}))}$ to the q-Cartan matrix of $\mathcal{P}_n$.

Proof – By Lemma 6.1 and Theorem 6.2, for any $\nu, \mu \in P_n^{(2)}$ we have

$$(107) \qquad\qquad e_\nu^{(2)} * \mathcal{P}_n * e_\mu^{(2)} \cong e_\nu * \mathbf{BSym}_n * e_\mu,$$

which proves (i). For the first part of (ii), we recall that a combinatorial description of the Cartan invariants of $\mathbf{BSym}_n$ was given in [8]. If we restrict that description to 2-peak partitions, then the resulting formulation is equivalent to that described in Conjecture 5.5. The final assertion follows from the description of the quiver of $\mathcal{P}_n$ computed above. ∎

6.2. **Comparison with earlier works.** A basis of idempotents, and a complete set of orthogonal idempotents for $\mathcal{P}_n$, have been obtained in [5], relying on previous work on the descent algebras of type B [7]. Here is a short derivation of these results, using our realization of type B noncommutative symmetric functions [11].

Let us recall the notation

$$(108) \qquad\qquad \phi(t) = \log \sigma_t = \sum_{n \geq 1} \frac{\Phi_n}{n} t^n .$$

For $r = 2$,

$$(109) \qquad\qquad \sigma_t^\sharp := \sigma_t(A - q\bar{A})|_{q=-1} = \bar{\lambda}_t \sigma_t = \exp(\phi^\sharp(t)) .$$

We have already seen that

$$(110) \qquad\qquad \sigma_t^\sharp * \sigma_t^\sharp = (\sigma_t^\sharp)^2 = \exp(2\phi^\sharp(t)) ,$$

so that

$$(111) \qquad\qquad E_\lambda := \sum_{I \downarrow = \lambda} \frac{\Phi^{I\sharp}}{2^{l(I)} i_1 i_2 \cdots i_{l(I)}}$$

are orthogonal idempotents, summing to $\sigma_1^\sharp$.

Following [11, Def. 4.14], define

$$(112) \qquad\qquad \eta(t) := \sigma_t \cdot [\sigma_t^\sharp]^{-\frac{1}{2}} .$$

Then, $\eta(1)$ is an idempotent.

Denote by $\tilde{f} = f|_{\bar{A}=A}$ the projection onto $\mathbf{Sym}$ of an element of $\mathbf{MR}$. Then,

$$(113) \qquad\qquad \xi(t) := \tilde{\sigma}_t^\sharp = \lambda_t \sigma_t$$

satisfies

$$(114) \qquad\qquad \xi(-t) = \xi(t)^{-1} .$$

Indeed,

$$(115) \qquad\qquad \xi(-t) = \lambda_{-t} \sigma_{-t} = (\lambda_t \sigma_t)^{-1} ,$$

so that

$$(116) \qquad\qquad \tilde{\phi}^\sharp(-t) = -\tilde{\phi}^\sharp(t) ,$$

that is, only the ϕ_n with n odd survive the projection onto $\mathcal{P}$. Moreover,

$$
\begin{aligned}
(117) \qquad \tilde{\eta}(-t) &= \sigma_{-t}(\lambda_{-t}\sigma_{-t})^{-\frac{1}{2}} = \sigma_{-t}(\lambda_t\sigma_t)^{\frac{1}{2}} \\
&= [(\lambda_t\sigma_t)^{-\frac{1}{2}}\lambda_t\sigma_t\lambda_{-t}]^{-1} = [(\lambda_t\sigma_t)^{\frac{1}{2}}\lambda_{-t}]^{-1} = \tilde{\eta}(t)\,,
\end{aligned}
$$

so that $\tilde{\eta}(-t)$ is even, and only the η_{2k} survive. Hence, if $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) = (\lambda_0; \bar{\lambda})$ runs over partitions of n such that λ_0 is even (allowed to be 0) and the other parts are odd,

$$
(118) \qquad \tilde{E}_\lambda = \eta_{\lambda_0} \sum_{J\downarrow\bar{\lambda}} \frac{\tilde{\Phi}^{I\sharp}}{2^{l(I)}i_1 i_2 \cdots i_{l(I)}}
$$

is a complete set of orthogonal idempotents of $\mathcal{P}_n$, consisting of the nonzero images of a complete set for $\mathbf{BSym}_n$. These idempotents coincide with those of [5], but are different from ours. They have nevertheless a similar structure.

7. q-CARTAN INVARIANTS FOR GENERALIZED PEAK ALGEBRAS

Conjecture 5.5 provided a conjectural description of the Cartan invariants of the higher order peak algebras $\mathcal{P}_n^{(r)}$. The following presents their q-analogs: the coefficient of q^k in row λ, column μ, is the multiplicity of the simple module λ in the kth slice of the descending Loewy series of the indecomposable projective module μ. (Here q is an indeterminate, and not to be confused with the primitive root of unity that q denoted earlier.) The usual Cartan invariants can be obtained from this q-analog by setting $q = 1$. Note that the q-Cartan matrix also encodes the quiver of the algebra since the single powers of q in the matrix correspond to the arrows of the quiver.

Also note that for $r \geq n$, the q-Cartan matrix of $\mathcal{P}_n^{(r)}$ is the same, up to indexation, as the q-Cartan matrix of $\mathbf{Sym}_n$. Below we write partitions in the order opposite to the order $<$ defined above, and the zero entries of the matrices are represented by dots to enhance readability.

7.1. q-Cartan matrices for unital peak algebras ($r = 2$).

$$
(119) \qquad C_2^{(2)} = \begin{pmatrix} 1 & . \\ . & 1 \end{pmatrix}
$$

$$
(120) \qquad C_3^{(2)} = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix}
$$

$$
(121) \qquad C_4^{(2)} = \begin{pmatrix} 1 & q & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}
$$

$$
(122) \qquad C_5^{(2)} =
\begin{pmatrix}
1 & . & . & q & . & . \\
. & 1 & . & . & . & . \\
. & . & 1 & q & . & . \\
. & . & . & 1 & . & . \\
. & . & . & . & 1 & . \\
. & . & . & . & . & 1
\end{pmatrix}
$$

$$
(123) \qquad C_6^{(2)} =
\begin{pmatrix}
1 & . & q & q & . & q^2 & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & q & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & 1 & q & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{pmatrix}
$$

$$
(124) \qquad C_7^{(2)} =
\begin{pmatrix}
1 & . & . & . & q & q & . & . & q^2 & . & . \\
. & 1 & . & . & q & . & . & . & . & . & . \\
. & . & 1 & . & . & . & q & . & . & . & . \\
. & . & . & 1 & . & q & q & . & q^2 & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & q & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & q & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

$$
(125) \qquad C_8^{(2)} =
\begin{pmatrix}
1 & q & q & . & q & q & . & 2q^2 & q^2 & q^2 & . & q^3 & . & . \\
. & 1 & . & . & . & . & . & q & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & q & q & . & . & q^2 & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & q & . & . & . & . \\
. & . & . & . & . & . & 1 & . & q & q & . & q^2 & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & q & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & q & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

$$(126)\quad C_9^{(2)} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 2q & q & \cdot & \cdot & \cdot & \cdot & 2q^2 & q^2 & \cdot & \cdot & q^3 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & q & \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & q & \cdot & q & \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & \cdot & q & \cdot & \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & q & \cdot & q & q & \cdot & 2q^2 & q^2 & q^2 & \cdot & q^3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & q & \cdot & \cdot & q^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & q & \cdot & q^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}$$

7.2. q-Cartan matrices for unital peak algebras of order $r = 3$.

$$(127)\qquad C_3^{(3)} = \begin{pmatrix} 1 & q & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

$$(128)\qquad C_4^{(3)} = \begin{pmatrix} 1 & \cdot & \cdot & q & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & q & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$(129)\qquad C_5^{(3)} = \begin{pmatrix} 1 & \cdot & q & q & \cdot & q^2 & \cdot \\ \cdot & 1 & \cdot & q & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & q & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$
(130) \qquad C_6^{(3)} = \begin{pmatrix}
1 & q & q & . & q & q^2 & 2q^2 & . & q^3 & . \\
. & 1 & . & . & . & . & q & . & . & . \\
. & . & 1 & . & . & q & q & . & q^2 & . \\
. & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & 1 & . & . & q & . \\
. & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

$$
(131) \qquad C_7^{(3)} = \begin{pmatrix}
1 & . & q & . & . & q^2+q & q & q^2 & . & q^2 & q^3+q^2 & . & q^3 & . \\
. & 1 & . & . & . & q & . & . & q & . & q^2 & . & . & . \\
. & . & 1 & . & . & q & . & q & . & . & q^2 & . & . & . \\
. & . & . & 1 & . & q & q & . & q & q^2 & 2q^2 & . & q^3 & . \\
. & . & . & . & 1 & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & q & . & . & . \\
. & . & . & . & . & . & 1 & . & . & q & q & . & q^2 & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & q & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

(132)

$$
C_8^{(3)} = \begin{pmatrix}
1 & . & . & . & q & q & . & q^2+q & . & . & . & q^3+2q^2 & q^2 & q^3+q^2 & . & q^3 & q^4+q^3 & . & q^4 & . \\
. & 1 & . & . & . & . & . & . & . & . & . & q & . & . & . & . & q^2 & . & . & . \\
. & . & 1 & . & . & q & q & . & . & q & q^2 & . & q^2 & q^2 & . & q^3 & . & . & . & . \\
. & . & . & 1 & q & . & q & . & . & q & q^2 & . & 2q^2 & . & . & q^3 & . & . & . & . \\
. & . & . & . & 1 & . & . & q & . & . & q^2+q & q & q^2 & . & q^2 & q^3+q^2 & . & q^3 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & q & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & q & . & q & . & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & q & q & . & q & q^2 & 2q^2 & . & q^3 & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q & . & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

7.3. q-Cartan matrices for unital peak algebras of order $r = 4$.

$$(133) \qquad C_4^{(4)} = \begin{pmatrix} 1 & . & q & q^2 & . \\ . & 1 & . & . & . \\ . & . & 1 & q & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix}$$

$$(134) \qquad C_5^{(4)} = \begin{pmatrix} 1 & q & . & q^2 & q & q^2 & . \\ . & 1 & . & q & . & . & . \\ . & . & 1 & . & q & q^2 & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & q & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

$$(135) \qquad C_6^{(4)} = \begin{pmatrix}
1 & . & . & q & . & q^2+q & . & q^3 & q^2 & q^3 & . \\
. & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & 1 & . & q & . & q^2 & q & q^2 & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

$$(136) \quad C_7^{(4)} = \begin{pmatrix}
1 & . & q & q & q^2 & q & . & q^2 & q^3 & q^3+2q^2 & . & q^4+q^3 & q^3 & q^4 & . \\
. & 1 & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . & . \\
. & . & 1 & . & q & . & . & . & q^2 & q & . & q^2 & . & . & . \\
. & . & . & 1 & . & . & . & q & . & q^2+q & . & q^3 & q^2 & q^3 & . \\
. & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & q & . & q^2 & q & q^2 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

(137)

$$C_8^{(4)} = \begin{pmatrix}
1 & q & q & q & q^2 & . & q & 2q^2 & q^2 & . & 3q^3 & 2q^2 & q^2 & q^3 & 2q^4 & q^4+4q^3 & . & q^5+2q^4 & q^4 & q^5 & . \\
. & 1 & . & q & . & . & q & . & . & . & 2q^2 & q & . & . & q^3 & 2q^2 & . & q^3 & . & . & . \\
. & . & 1 & . & . & . & q & . & . & . & q^2+q & . & . & . & q^3 & q^2 & . & q^3 & . & . & . \\
. & . & . & 1 & . & . & q & q & . & . & q^2 & q & . & q^2 & q^3 & q^3+2q^2 & . & q^4+q^3 & q^3 & q^4 & . \\
. & . & . & . & 1 & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & q & . & . & . & q^2 & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & q & . & q^2+q & . & q^3 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & q^2 & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

7.4. q-Cartan matrices for unital peak algebras of order $r = 5$.

$$(138) \qquad C_5^{(5)} = \begin{pmatrix}
1 & q & q & q^2 & q^2 & q^3 & . \\
. & 1 & . & q & . & . & . \\
. & . & 1 & . & q & q^2 & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & 1 & q & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & 1
\end{pmatrix}$$

$$(139) \qquad C_6^{(5)} = \begin{pmatrix}
1 & . & q & . & . & q^2+q & q & q^3 & q^2 & q^3 & . \\
. & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

$$(140) \qquad C_7^{(5)} = \begin{pmatrix}
1 & q & . & q & q & q^2 & 2q^2 & . & q^2 & 3q^3 & q^2 & 2q^4 & q^3 & q^4 & . \\
. & 1 & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . & . \\
. & . & 1 & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . & . \\
. & . & . & 1 & . & . & q & . & . & q^2+q & q & q^3 & q^2 & q^3 & . \\
. & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

(141)

$$
C_8^{(5)} = \begin{pmatrix}
1 & \cdot & \cdot & q & q & q & q^2 & q^2{+}q & \cdot & q^2 & \cdot & q^3{+}2q^2 & q^3 & 2q^3{+}q^2 & \cdot & q^4{+}q^3 & 3q^4{+}q^3 & q^3 & 2q^5 & q^4 & q^5 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & q^2 & q^2 & \cdot & \cdot & q^3 & \cdot & q^4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & q & q & \cdot & \cdot & 2q^2 & q^2 & q^2 & \cdot & q^3 & 2q^3 & \cdot & q^4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & q^2{+}q & \cdot & q & \cdot & q^3 & q^2 & \cdot & q^3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & q & \cdot & q & q^2 & 2q^2 & \cdot & q^2 & 3q^3 & q^2 & 2q^4 & q^3 & q^4 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & q & \cdot & \cdot & 2q^2 & \cdot & q^3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & \cdot & \cdot & q^2 & q^2 & \cdot & q^3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & q^2{+}q & q & q^3 & q^2 & q^3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & \cdot & q^2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q & q & q^2 & q^2 & q^3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & q^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

7.5. q-Cartan matrices for unital peak algebras of order $r = 6$.

(142)

$$
C_6^{(6)} = \begin{pmatrix}
1 & \cdot & q & q & \cdot & 2q^2 & q^2 & q^3 & q^3 & q^4 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & q & \cdot & q^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & q & \cdot & q^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & q & q & q^2 & q^2 & q^3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & q & q^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

$$(143) \qquad C_7^{(6)} = \begin{pmatrix}
1 & q & q & . & q^2 & q^2 & 2q^2 & q & q^3 & 3q^3 & q^2 & 2q^4 & q^3 & q^4 & . \\
. & 1 & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . & . \\
. & . & 1 & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . & . \\
. & . & . & 1 & . & . & q & q & . & 2q^2 & q^2 & q^3 & q^3 & q^4 & . \\
. & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

(144)

$$C_8^{(6)} = \begin{pmatrix}
1 & . & q & . & q & q^2 & q & 2q^2 & 2q^2 & . & . & 3q^3 & q^3 & 3q^3 & q^2 & 2q^4 & 4q^4 & q^3 & 2q^5 & q^4 & q^5 & . \\
. & 1 & . & . & . & . & . & q & . & . & . & . & q^2 & q^2 & . & . & q^3 & . & q^4 & . & . & . \\
. & . & 1 & . & . & q & . & q & q & . & . & 2q^2 & q^2 & q^2 & . & q^3 & 2q^3 & . & q^4 & . & . & . \\
. & . & . & 1 & . & . & q & . & q & . & . & 2q^2 & . & q^2 & . & q^3 & q^3 & . & q^4 & . & . & . \\
. & . & . & . & 1 & . & . & q & q & . & . & q^2 & q^2 & 2q^2 & q & q^3 & 3q^3 & q^2 & 2q^4 & q^3 & q^4 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & q & q & . & 2q^2 & q^2 & q^3 & q^3 & q^4 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & q & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

7.6. q-Cartan matrices for unital peak algebras of order $r = 7$.

$$(145) \quad C_7^{(7)} = \begin{pmatrix}
1 & q & q & q & q^2 & q^2 & 2q^2 & q^2 & q^3 & 3q^3 & q^3 & 2q^4 & q^4 & q^5 & . \\
. & 1 & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . & . \\
. & . & 1 & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . & . \\
. & . & . & 1 & . & . & q & q & . & 2q^2 & q^2 & q^3 & q^3 & q^4 & . \\
. & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

(146)

$$C_8^{(7)} = \begin{pmatrix}
1 & . & q & q & . & q^2 & q^2 & q^2+q & 2q^2 & q & . & 3q^3 & q^3 & 2q^3+q^2 & q^2 & 2q^4 & 3q^4+q^3 & q^3 & 2q^5 & q^4 & q^5 & . \\
. & 1 & . & . & . & . & . & q & . & . & . & . & q^2 & q^2 & . & . & q^3 & . & q^4 & . & . & . \\
. & . & 1 & . & . & q & . & q & q & . & . & 2q^2 & q^2 & q^2 & . & q^3 & 2q^3 & . & q^4 & . & . & . \\
. & . & . & 1 & . & . & q & . & q & . & . & 2q^2 & . & q^2 & . & q^3 & q^3 & . & q^4 & . & . & . \\
. & . & . & . & 1 & . & . & q & q & q & . & q^2 & q^2 & 2q^2 & q^2 & q^3 & 3q^3 & q^3 & 2q^4 & q^4 & q^5 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & q & . & q & . & 2q^2 & . & q^3 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & q & q & . & 2q^2 & q^2 & q^3 & q^3 & q^4 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & q^2 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

7.7. q-Cartan matrices for unital peak algebras of order $r = 8$.

(147)

$$C_8^{(8)} = \begin{pmatrix}
1 & . & q & q & q & q^2 & q^2 & 2q^2 & 2q^2 & q^2 & . & 3q^3 & q^3 & 3q^3 & q^3 & 2q^4 & 4q^4 & q^4 & 2q^5 & q^5 & q^6 & . \\
. & 1 & . & . & . & . & . & q & . & . & . & . & q^2 & q^2 & . & . & q^3 & . & q^4 & . & . & . \\
. & . & 1 & . & . & q & . & q & q & . & . & 2q^2 & q^2 & q^2 & . & q^3 & 2q^3 & . & q^4 & . & . & . \\
. & . & . & 1 & . & . & q & . & q & . & . & 2q^2 & . & q^2 & . & q^3 & q^3 & . & q^4 & . & . & . \\
. & . & . & . & 1 & . & . & q & q & q & . & q^2 & q^2 & 2q^2 & q^2 & q^3 & 3q^3 & q^3 & 2q^4 & q^4 & q^5 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & q & . & . & . & q^2 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & q & q & . & . & 2q^2 & . & q^3 & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & q & . & q & . & q^2 & q^2 & . & q^3 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & q & q & . & 2q^2 & q^2 & q^3 & q^3 & q^4 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & q & . & q^2 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q & q^2 & q^2 & q^3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & q & q^2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & q & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}$$

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